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THE LEONOV–PANASYUK–DUGDALE MODEL FOR A CRACK AT THE INTERFACE OF THE JOINT OF MATERIALS†

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The plane problem of the limit equilibrium of a crack – delamination at the interface of different materials is considered. It is assumed that constant normal and shear stresses of cohesion act between the crack surfaces in the end regions when uniformly distributed normal stresses are applied at infinity. The general case when the dimensions of the end regions are not small compared with the characteristic size of the crack is investigated. Analytic expressions are obtained for the components of the vector of the crack opening, the stress distribution a head of the crack, the stress intensity factors and, also, the relations between the external load, the length of the crack and the end region parameters in the limit equilibrium state. The case when the end regions are small compared with the length of the crack is considered separately.

Estimation of the resistance of adhesive joint to the growth of an interface crack implies some type of modelling of the deformation process and preparations for rupture in the end regions of crack delaminations taking into account, the mechanical properties and geometrical characteristics of the intermediate adhesive layer.

The model of a crack [1, 2] at the interface of two media with bonds acting between their surfaces in the end regions of the crack can be considered for this purpose. The case of linearly deformable bonds has been considered in detail in [3]. It is assumed in the development of the models of Barenblatt [4–6] for cracks in homogeneous bodies and of Salganik [7] for interface cracks that the end regions, which are occupied by the bonds, are not necessarily small compared with the size of the crack. The problem of crack delamination under the action of a normal external load and of the normal and shear stresses in the bonds which are impeding the opening of the crack reduces to a system of singular integro-differential equations with kernels of the Cauchy type which is solved numerically.

At the same time, it is of interest to construct a model which, on the one hand, enables one to take account of the processes in the end regions of the crack at the boundary of the two media and, on the other hand, the analytic solution of the problem of the limit equilibrium of a crack. Such a model is proposed in this paper and is, in essence, an extension of the Leonov–Panasyuk–Dugdale model [8, 9] for cracks in homogeneous media. It is assumed that, under the action of uniformly distributed normal stresses at infinity, the interaction between the surfaces in the end regions of the crack is characterized by constant normal and shear cohesive stresses. In particular, this assumption enables one to model the plastic flow in the intermediate adhesive layer in the end regions of the crack.

We note that, whereas in homogeneous bodies, the hypothesis of a thin plastic zone at the crack tip provides a good description of crack growth processes in thin plates, the plastic zones, in the case of a piecewise–homogeneous body when the thickness of the adhesive layer is small compared with the length of the crack, can be localized within the limits of it and not only under the conditions for a plane stressed state. This certainly holds if the adhesive layer is more plastic than the materials which are joined by it.

We also note that, although the solving boundary-value problem of elasticity, to which the search for the opening of an interface crack reduces within the framework of the proposed model, has, in the simplest version of half-planes as well as in the case of crack – delamination without plastic zones, oscillatory singularities close to the crack tips, this does not prevent one from obtaining interesting relations between the end-region parameters, the length of the crack and the external load at its limit equilibrium. This last point is associated with the fact that the boundary-value problem being considered

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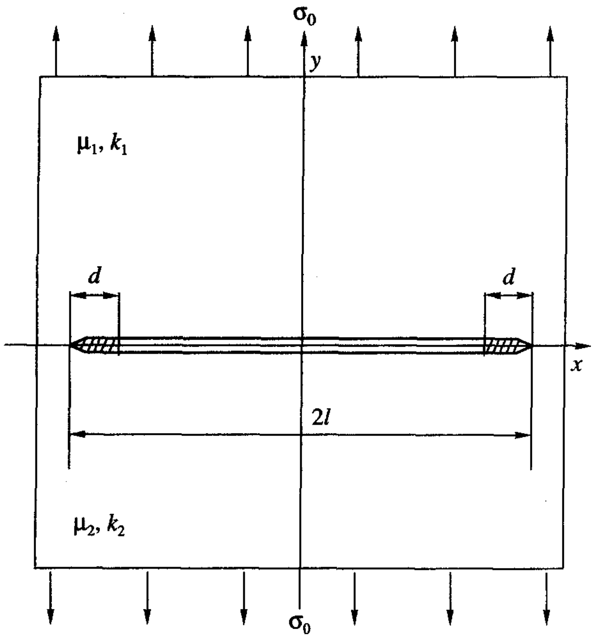

Fig. 1

is in essence outer with respect to the initial problem of a crack when there is an adhesive layer, the existence of which in the inner problem excludes the non-physical need for the overlapping of the crack surfaces in the case of an oscillatory singularity in the solution close to the crack tip.

The possibilities of modifying the formulation of the problem of an interface crack which enable one to remove the oscillatory singularity by introducing contact zones with frictionless slipping [10–13] or with a limit friction [14] have been specially investigated in a number of papers [10–14].

1. FORMULATION OF THE PROBLEM.

Basic equations. Consider the plane problem of a crack of length $2l$, located at the interface of two unbounded, elastic half-planes made of different materials $|x| \le l, y = 0$. Suppose a medium with the parameters $\mu = \mu_1, k = k_1$ occupies the upper half-plane ($y > 0$), and a medium with the parameters $\mu = \mu_2, k = k_2$ fills the lower half-plane ($y < 0$), where $k_{1,2} = 3 - 4\nu_{1,2}$ for plane stress and $k_{1,2} = (3 - \nu_{1,2})/(1 + \nu_{1,2})$ for a plane stress, and $\mu_1 = \mu_2$ and $\nu_1 = \nu_2$ are the shear moduli and Poisson's ratios of the materials. We will assume that uniformly distributed normal stresses σ_0 are imposed on the far boundary ($x^2 + y^2 \to \infty$), while constant normal and shear adhesive stresses between the edges of the crack surfaces σ_* and τ_* (Fig. 1) act in the end regions of length d adjacent to the crack tips ($l - d \le |x| \le l, y = 0$). These adhesive stresses correspond to the plastic flow of the adhesive in a thin intermediate layer and satisfy a certain plasticity criterion:

$$f(\sigma_*, \tau_*) = 0$$

where f is a monotonically increasing function of the absolute values of σ_* and τ_* which depends on the adhesive properties. We note that the condition that the stresses are bounded at the crack tip at $x = \pm l$ (see below) gives a second relation for determining the values of σ_* and τ_*.

Using the linearity of the problem, the state being considered can be represented by a superposition of the following two states: 1) the adhesive joint of the materials without a crack under the action of a constant tensile stress σ_0 at infinity, and 2) adhesive joint of the materials with a crack at the interface in which case the tensile stress σ_0 is removed at the crack surfaces.

We introduce the dimensionless quantities

$$x' = x/l, \quad d' = d/l$$

Omitting the primes and taking account of what has been said above, we shall consider the problem of a crack of unit half-length with the following boundary conditions on its surfaces

$$\sigma_{yy}(x) = \begin{cases} -\sigma_0+\sigma_*, & 1-d \le |x| \le 1 \\ -\sigma_0, & |x| < 1-d \end{cases}, \quad \sigma_{xy}(x) = \begin{cases} \tau_*, & 1-d \le x \le 1 \\ 0, & |x| < 1-d \\ -\tau_*, & -1 \le |x| \le -(1-d) \end{cases} \tag{1.1}$$

We consider the dimensionless (with respect to the l) vector of the crack opening

$$\mathbf{u}(u_x(x), u_y(x)) = \mathbf{u}^+(u_x^+(x), u_y^+(x)) - \mathbf{u}^-(u_x^-(x), u_y^-(x)) \tag{1.2}$$

where $\mathbf{u}^+(u_x^+(x), u_y^+(x))$ and $\mathbf{u}^-(u_x^-(x), u_y^-(x))$ are vectors of the dimensionless displacements of the upper and lower crack surfaces. We obtain the derivatives with respect to the x of the components of the vector of the crack opening, when there are arbitrary normal $\sigma_{yy}(x)$ and shear $\sigma_{xy}(x)$ stress on the surfaces of a crack of length 2 located at the interface of the materials, following an approach which has been described previously in [3]:

$$u_y'(x) - iu_x'(x) = \frac{\gamma}{4(1+\alpha)}\left[(1-\alpha)(\sigma_{xy}(x)+i\sigma_{yy}(x)) - \right.$$

$$\left. - \frac{1+\alpha}{\pi\sqrt{1-x^2}}\left(\frac{1-|x|}{1+|x|}\right)^{-i\beta\,\mathrm{sign}\,x} \int_{-1}^{1} \frac{(\sigma_{yy}(\xi) - i\sigma_{xy}(\xi))d\xi}{(\xi - x)\varphi(\xi)}\right] \tag{1.3}$$

where

$$i^2 = -1, \quad \alpha = \frac{\mu_2 k_1 + \mu_1}{\mu_1 k_2 + \mu_2}, \quad \beta = \frac{\ln\alpha}{2\pi}, \quad \gamma = \frac{k_1+1}{\mu_1} + \frac{k_2+1}{\mu_2}$$

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\left(\frac{1-x}{1+x}\right)^{-i\beta}$$

Substituting expressions (1.1) into equality (1.3), we obtain

$$u_y'(x) - iu_x'(x) = \frac{\gamma\varphi(x)}{4}\left[\frac{2i\beta - x}{\mathrm{ch}\pi\beta}\sigma_0 + (\sigma_* - i\tau_*)Q_1(x) - (\sigma_* + i\tau_*)Q_2(x)\right] \tag{1.4}$$

where

$$Q_1(x) = \begin{cases} R(x, 2/d - 1, \beta) - (2i\beta - x)/\mathrm{ch}\pi\beta, & 1-d < x < 1 \\ R\left(-x, \frac{1}{2/d-1}, -\beta\right), & 0 < x < 1-d \end{cases}$$

$$Q_2(x) = R\left(x, \frac{1}{2/d-1}, \beta\right) \tag{1.5}$$

$$R(x_1, x_2, \beta) = \frac{2x_2^{1/2-i\beta}}{\pi(1-2i\beta)}\left\{\frac{1}{1+x_2}\left[1 - 2i\beta + (-x_1 + 2i\beta)F\left(1, 1, 3/2 - i\beta, \frac{x_2}{1+x_2}\right)\right] - \right.$$

$$\left. - (1-x_1)F\left(1, 1/2 - i\beta, 3/2 - i\beta, \frac{1-x_1}{1+x_1}x_2\right)\right\}$$

The hypergeometric function of the complex variable z which, in the circle $|z| < 1$, is represented by the series

$$F(a, b, c, z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k, \quad (\lambda)_k = \lambda(\lambda+1)\ldots(\lambda+k-1), \quad (\lambda)_0 = 1$$

$$c \ne 0, \quad -1, -2, -3, \ldots$$

and, in the domain which does not belong to the circle of convergence, is determined by an analytic extension of this series, is denoted by $F(a, b, c, z)$. The function $F(a, b, c, z)$ is regular in the complex plane with a cut $(1, \infty)$.

From the condition that the vector of the crack opening is equal to zero when $x = 1$

$$u_x(1) = 0, \quad u_y(1) = 0 \tag{1.6}$$

and, using the formula

$$u_y(x) - iu_x(x) = \int_1^x (u'_y(\xi) - iu'_x(\xi))d\xi$$

we find, for $1 - d \le x \le 1$, the components of the vector of the crack opening

$$u_y(x) - iu_x(x) = \frac{\gamma}{4\operatorname{ch}\pi\beta}(1 - x^2)\varphi(x)[\sigma_0 - (\sigma_* - i\tau_*)P_1(x) - (\sigma_* + i\tau_*)P_2(x)] \tag{1.7}$$

where

$$P_1(x) = 1 - S(x, 2/d - 1, \beta), \quad P_2(x) = S\left(x, \frac{1}{2/d - 1}, \beta\right)$$

$$S(x_1, x_2, \beta) = \frac{2\operatorname{ch}\pi\beta x_2^{1/2 - i\beta}}{\pi(1 - 2i\beta)(1 + x_2)}\times \tag{1.8}$$

$$\times\left[F\left(1, 1, 3/2 - i\beta, \frac{x_2}{1 + x_2}\right) - F\left(1, 1, 3/2 - i\beta, \frac{(1 - x_1)x_2}{(1 - x_1)x_2 - (1 + x_1)}\right)\right]$$

In the case of values $0 \le x \le 1 - d$, the components of the vector of the crack opening are determined taking account of the continuity of this opening vector when $x = 1 - d$ using the formula

$$u_y(x) - iu_x(x) = \int_{1-d}^x (u'_y(\xi) - iu'_x(\xi))d\xi + u_y(1 - d) - iu_x(1 - d) \tag{1.9}$$

The derivatives with respect to the x of the components of the vector of the crack opening in the integrand are chosen in accordance with expression (1.4) and the components of the opening vector at the edge of the end region $x = 1 - d$ are determined using formula (1.7). We have

$$u_y(1 - d) - iu_x(1 - d) = \gamma\left\{(1 - d/2)^{1/2 + i\beta}(d/2)^{1/2 - i\beta}\frac{\sigma_0}{2\operatorname{ch}\pi\beta} - \right.$$

$$-\frac{1}{\pi}\left[(1 - d/2)d/2F(1, 1, 3/2 + i\beta, d/2)\frac{\sigma_* - i\tau_*}{1 + 2i\beta} + (1 - d/2)^{1 + 2i\beta}(d/2)^{1 - 2i\beta}\times\right. \tag{1.10}$$

$$\left.\left.\times\left(F(1, 1, 3/2 - i\beta, d/2) - F\left(1, 1, 3/2 - i\beta, \frac{-1}{(2/d - 1)^2 - 1}\right)\right)\frac{\sigma_* + i\tau_*}{1 - 2i\beta}\right]\right\}$$

and, for $0 \le x \le 1 - d$, we find

$$u_y(x) - iu_x(x) = \frac{\gamma}{4\operatorname{ch}\pi\beta}(1 - x^2)\varphi(x)\times$$

$$\times\left[\sigma_0 - (\sigma_* - i\tau_*)S\left(-x, \frac{1}{2/d - 1}, -\beta\right) - (\sigma_* + i\tau_*)S\left(x, \frac{1}{2/d - 1}, \beta\right)\right] \tag{1.11}$$

The function $S(x_1, x_2, \beta)$ is defined by formula (1.8).

The components of the displacement vector of the upper and lower crack surface and the crack opening vector are connected by the following relations [15]

$$\frac{u_y^-(x)}{u_y^+(x)} = \frac{u_x^-(x)}{u_x^+(x)} = -\frac{\mu_1(k_2\alpha+1)}{\mu_2(k_1+\alpha)} \tag{1.12}$$

Then, using the opening vector $u(u_x(x), u_y(x))$, we find the displacement vector of the upper and lower crack surface $u^+(u_x^+(x), u_y^+(x))$ and $\mathbf{u}^-(u_x^-(x), u_y^-(x))$ using the formulae

$$\mathbf{u}^+ = \mathbf{c}^+\mathbf{u}, \quad \mathbf{u} = \mathbf{c}^-\mathbf{u} \tag{1.13}$$

where

$$c^+ = \left[1+\frac{\mu_1(k_2\alpha+1)}{\mu_2(k_1+\alpha)}\right]^{-1}, \quad c^- = -\left[1+\frac{\mu_2(k_1+\alpha)}{\mu_1(k_2\alpha+1)}\right]^{-1}$$

The stress distribution ahead of the crack ($x > 1$) in the case of an arbitrary distribution of the normal stresses $\sigma_{yy}(x)$ and shear stresses $\sigma_{xy}(x)$ on the surfaces of a crack of length 2 has the form [15]

$$\sigma_{xy}(x) + i\sigma_{yy}(x) = \frac{\mathrm{ch}\pi\beta}{\pi}\psi(x)\int_{-1}^{1}\frac{(\sigma_{xy}(\xi)+i\sigma_{yy}(\xi))}{(\xi-x)\varphi(\xi)}d\xi \tag{1.14}$$

where

$$\psi(x) = \frac{1}{\sqrt{x^2-1}}\left(\frac{x-1}{x+1}\right)^{-i\beta}$$

Substituting expression (1.1) into equality (1.14), we obtain

$$\begin{aligned}\sigma_{xy}(x) + i\sigma_{yy}(x) &= i\sigma_0[-1+(x-2i\beta)\psi(x)] - \\ &- i\,\mathrm{ch}\pi\beta\psi(x)\left[(\sigma_* - i\tau_*)R\left(-x, \frac{1}{2/d-1}, -\beta\right) - (\sigma_* + i\tau_*)R\left(x, \frac{1}{2/d-1}, \beta\right)\right]\end{aligned} \tag{1.15}$$

The function $R(x_1, x_2, \beta)$ is defined by formula (1.5).

For a crack of length $2l$, located at the interface of two materials, the expression for the stress intensity factors (SIFs)

$$\begin{aligned}K_{\mathrm{I}} + iK_{\mathrm{II}} &= l^{1/2-i\beta}\lim_{s=x-1\to+0}\sqrt{2\pi s}(\sigma_{yy}(s)+i\sigma_{xy}(s))s^{-i\beta} = \\ &= \frac{-\mathrm{ch}\pi\beta}{2^{1/2+i\beta}}\sqrt{\frac{2}{\pi}}l^{1/2-i\beta}\int_{-1}^{1}\left(\frac{1+\xi}{1-\xi}\right)^{1/2+i\beta}(\sigma_{yy}(\xi)+i\sigma_{xy}(\xi))d\xi\end{aligned} \tag{1.16}$$

holds in the case of an arbitrary distribution of the normal and shear stresses.

Substituting expressions (1.1) into (1.16), we obtain the total SIFs from the action of the external tensile stresses and the internal adhesive stresses in the end regions of the crack

$$\begin{aligned}K_{\mathrm{I}} + iK_{\mathrm{II}} &= \frac{1+2i\beta}{2^{i\beta}}\sqrt{\pi}\sigma_0 l^{1/2-i\beta} - \frac{\mathrm{ch}\pi\beta}{2^{i\beta}\sqrt{\pi}}l^{1/2-i\beta}\left[(\sigma_* - i\tau_*)\frac{3/2-i\beta}{2(9/4+\beta^2)}\times\right. \\ &\times d^{3/2+i\beta}(2-d)^{1/2-i\beta}F(1, 2, 5/2+i\beta, d/2) + \\ &\left. + (\sigma_* + i\tau_*)\frac{1+2i\beta}{1+4\beta^2}d^{1/2-i\beta}(2-d)^{3/2+i\beta}F(1, 2, 3/2-i\beta, d/2)\right]\end{aligned} \tag{1.17}$$

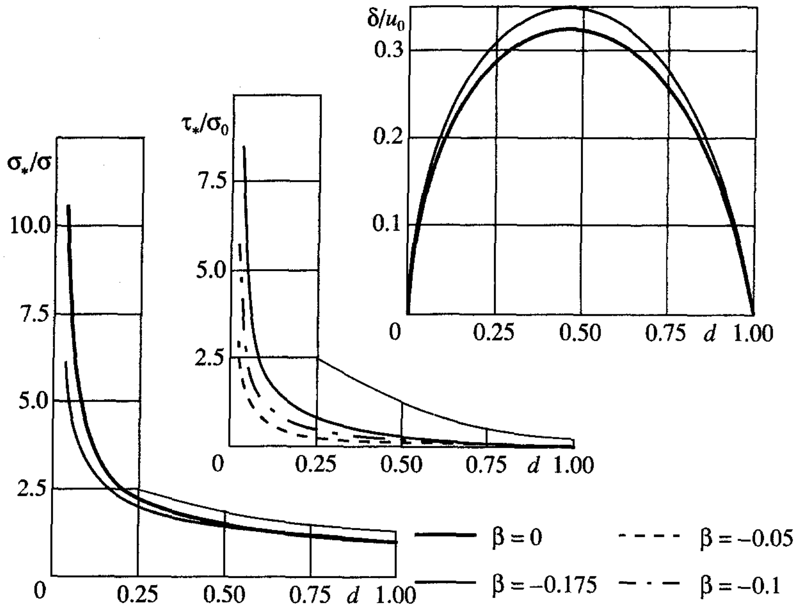

Fig. 2

2. ANALYSIS OF THE LIMIT EQUILIBRIUM STATE

We will now consider the limit equilibrium state of crack – delamination at the interface of the different materials, which is characterized by the action of the adhesive forces in the end region of the crack and for which there is no energy flux through the points $x = \pm 1$ accompanying the crack growth. The case when the total SIFs are equal to zero corresponds to this state. Equating expression (1.17) to zero, we obtain a relation which connects the external load, the adhesive stresses and the size of the end region of the crack in the limit equilibrium state. The limit equilibrium state is described by the following formulae

$$\frac{\sigma_*}{\sigma_0} = \frac{\pi}{\Delta}\cos(\beta\ln(2/d-1)),\quad \frac{\tau_*}{\sigma_0} = -\frac{\pi}{\Delta}\sin(\beta\ln(2/d-1))$$
$$\Delta = 2\operatorname{ch}\pi\beta(d(2-d))^{1/2}\operatorname{Re}\left[\frac{1}{1+2i\beta}F(1,1,3/2+i\beta,d/2)\right] \tag{2.1}$$

whence it also follows that

$$\frac{\tau_*}{\sigma_*} = -\operatorname{tg}(\beta\ln(2/d-1))$$

The values of the limit load σ_0 and the adhesive stresses σ_*, τ_* can be determined for specified values of β and d from relations (2.21) and the criterion of the plastic flow of the material the thin intermediate adhesive layer. If, in accordance with expression (2.1), σ_*, τ_* is substituted into the plastic flow criterion $f(\sigma_*, \tau_*) = 0$, it is possible to construct the function d from σ_0/σ_{*T} (the value of σ_{*T}, which is determined from the plastic flow criterion $f(\sigma_{*T}, 0) = 0$, is the yield point of the adhesive) and to compare it with the experimental curve, which can be constructed using measurements of the size of the plastic zone d and the limit load σ_0 as was done in [9] in the case of cracks in homogeneous materials.

Graphs of the ratios of the normal adhesive stress σ_* and the shear adhesive stress τ_* to the load σ_0 against the dimensionless length of the end region d for various values of the parameter β, constructed using formulae (2.1), are shown in Fig. 2. (Note that, since the parameter α lies within the limits $1/3 < \alpha < 3$, the parameter β lies within the limits $-0.175 < \beta < 0.175$.) In particular, in the case of a fixed ratio d, the graphs demonstrate a redistribution of the normal adhesive stress σ_* and the shear adhesive stress τ_* depending on the change in the value of the parameter β.

Substituting the normal adhesive stress σ_* and the shear adhesive stress τ_* according to formula (2.1) into equality (1.7), we obtain the components of the vector the crack opening in the end region

$1-d \le x \le 1$. Next, from the condition $u_y(x) > 0$, we determine that the crack opening is positive in the interval $|x| < 1-\varepsilon$, where the parameter ε depends on β and d. In the intervals $1-\varepsilon < |x| < 1$, the crack opening has an oscillatory singularity and takes negative values an unlimited number of times. It is therefore necessary, for the model being considered to be applicables that the size of the end region should be significantly greater than the domain in which the solution oscillates, that is

$$d \gg \varepsilon(\beta, d) \tag{2.2}$$

In the case of real adhesive joints with cracks at the interface, this condition is well satisfied. Consider, for example, a combination of steel and plexiglass (Young's modulus and Poisson's ratio for steel are $E_1 = 210$ GPa and $\nu_1 = 0.3$, and for plexiglass are $E_2 = 3$GPa and $\nu_2 = 0.4$) with an epoxy adhesive layer of thickness $h \sim 0.1$ mm [16]. Then, for $l \sim 1$ cm and $d \sim h/l$, we have: $d \sim 10^{-2}$ and, in the case of plane strain ($\beta \approx -0.05$), we obtain $\varepsilon \sim 10^{-12}$ and, in the case of a plane stress ($\beta \approx -0.1$), we obtain $\varepsilon \sim 10^{-6}$.

The fact that the total SIFs are equal to zero (1.17) leads to the boundedness of the stresses close to the crack tip. Taking account of relation (2.1), we obtain the stress distribution ahead of the crack ($x > 1$)

$$\sigma_{xy}(x) + i\sigma_{yy}(x) = -i\sigma_0 + i\sigma_0\frac{\pi}{\Delta}(2/d-1)^{i\beta}\left\{1 - \frac{2\operatorname{ch}\pi\beta}{\pi(1-2i\beta)}\times\right.$$

$$\times\left(\frac{x-1}{x+1}(2/d-1)\right)^{1/2-i\beta}\left[F\left(1, 1/2-i\beta, 3/2-i\beta, -\frac{x-1}{x+1}(2/d-1)\right)-\right.$$

$$\left.\left.-\frac{1}{2/d-1}F\left(1, 1/2-i\beta, 3/2-i\beta, -\frac{x-1}{x+1}\frac{1}{2/d-1}\right)\right]\right\} \tag{2.3}$$

The condition of limit stretching δ in the edge of the end region, where δ is a constant which characterizes the material of the adhesive layer, is one of the conditions used to estimate the size of the end region d. We shall use the dimensionless quantity $\delta' = \delta/l$, henceforth omitting the prime. We then have

$$|u_y(1-d) - iu_x(1-d)| = \delta \tag{2.4}$$

Substituting the expressions for the normal adhesive stress σ_* and the shear adhesive stress τ_* (2.1) into formulae (1.10) and then using condition (2.4), we obtain a relation which connects the limit stretching, the load, the length of the crack and the size of the end region in the limit equilibrium state. Graphs of the limit stretching, normalized to the opening at the centre of the crack

$$u_0 = \frac{\gamma}{4\operatorname{ch}\pi\beta}\sigma_0$$

for the case of a zero end region against the parameter d for the minimum and maximum values (in absolute magnitude) of the parameter β are shown in Fig. 2.

When the materials are the same ($\mu_1 = \mu_2 = \mu$, $k_1 = k_2 = k$, $\beta = 0$), expressions (2.1), (2.3) and (2.4) become relations which are analogous to the relations of the Leonov–Panasyuk–Dugdale model for cracks in homogeneous media

$$\frac{\sigma_*}{\sigma_0} = \frac{\pi}{2\arccos(1-d)}, \quad \tau_* = 0$$

$$\sigma_{yy}(x) = -\sigma_0 + \frac{\sigma_0}{\arccos(1-d)}\operatorname{arctg}\sqrt{\frac{(1-d)^{-2}-1}{1-x^{-2}}}, \quad \sigma_{xy}(x) = 0$$

$$\frac{-(1-d)\ln(1-d)}{\arccos(1-d)}\frac{k+1}{2\mu}\sigma_0 = \delta$$

In these relations, the value of σ_*, which is determined from the plasticity criterion $f(\sigma_*, 0) = 0$, has the meaning of the yield point of the adhesive.

3. THE CASE OF A SMALL PARAMETER d

We will now take the limit in the relations which have been obtained above when the size of the end region is small compared with the length of the crack. From expression (1.17), we have the leading term of the asymptotic expansion when $d \to 0$

$$K_{\mathrm{I}} + iK_{\mathrm{II}} = \frac{1+2i\beta}{2^{i\beta}}\sqrt{\pi}\sigma_0 l^{1/2-i\beta} - 2^{3/2+i\beta}\frac{1+2i\beta}{1+4\beta^2}\frac{\operatorname{ch}\pi\beta}{2^{i\beta}\sqrt{\pi}} d^{1/2-i\beta} l^{1/2-i\beta}(\sigma_* + i\tau_*) \tag{3.1}$$

Equating this expression to zero, we obtain the relation

$$\sigma_* + i\tau_* = \frac{\pi(1+4\beta^2)}{2\sqrt{2}\operatorname{ch}\pi\beta\sqrt{d}}(d/2)^{i\beta}\sigma_0 \tag{3.2}$$

which connects the external load, the adhesive stresses, and the length and the size of the end region of the crack in the limit equilibrium state.

The components of the vector for the crack opening at the edge of the end region $x = 1-d$ are determined by formula (1.10). The leading term of the asymptotic expansion when $d \to 0$ is equal to

$$u_y(1-d) - iu_x(1-d) = \gamma\left[\frac{\sqrt{d/2}(d/2)^{-i\beta}}{2\operatorname{ch}\pi\beta}\sigma_0 - \frac{d}{2\pi(1+2i\beta)}(\sigma_* - i\tau_*)\right] \tag{3.3}$$

Using relation (3.2), we obtain

$$u_y(1-d) - iu_x(1-d) = \frac{\gamma d}{2\pi}\frac{1+2i\beta}{1+4\beta^2}(\sigma_* - i\tau_*) \tag{3.4}$$

In the case of the limit stretching condition (2.4) we obtain

$$d = \frac{2\pi\sqrt{1+4\beta^2}}{\gamma\sqrt{\sigma_*^2 + \tau_*^2}}\delta \tag{3.5}$$

for the size of the end region.

We will now determine the critical magnitude of the rate of absorption of energy at crack tip, which is required for the crack to grow. For the elastic energy release rate accompanying the crack advance, we have the expression [15]

$$G = \frac{\gamma K_0^2}{16\operatorname{ch}^2\pi\beta} \tag{3.6}$$

where $K_0 = \sqrt{(K_{\mathrm{I}0})^2 + (K_{\mathrm{II}0})^2}$ is the modulus of the SIF due to the action of the external load σ_0. The equality

$$G = G_*$$

holds in the limit equilibrium state. At the same time, the critical value of the rate of energy absorption at the crack tip is given by the formula

$$G_* = \frac{\gamma K_*^2}{16\operatorname{ch}^2\pi\beta}$$

where $K_* = \sqrt{(K_{\mathrm{I}*})^2 + (K_{\mathrm{II}*})^2}$ is the modulus of the SIF due to the adhesive stresses acting in the end region of the crack in the limit equilibrium state.

From relation (3.1), we find

$$K_0^2 = \pi(1+4\beta^2)\sigma_0^2 l, \quad K_*^2 = \frac{8\operatorname{ch}^2\pi\beta}{\pi(1+4\beta^2)}(\sigma_*^2 + \tau_*^2)dl$$

We then obtain

$$G_* = \frac{\gamma}{2\pi(1+4\beta^2)}(\sigma_*^2 + \tau_*^2)dl \tag{3.7}$$

In the case of the limit stretching condition, the relation

$$G_* = \sqrt{\frac{\sigma_*^2 + \tau_*^2}{1+4\beta^2}}\delta l \approx \sqrt{\sigma_*^2 + \tau_*^2}\delta l \tag{3.8}$$

follows from expressions (3.5) and (3.7), which associates the magnitude of the rate of energy absorption (the adhesion fracture energy) with the stresses acting in the end regions and with the limit stretching.

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